Midnapore College (Autonomous)

Department of Statistics (4-TH Semester)

Solve the problems for paper 401

Problems on Most Powerful Test

1. (a) Let X_1, X_2, X_3, X_4, X_5 be a random sample from $N(2, \sigma^2)$ distribution, where σ^2 is unknown. Derive the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 4$ against $\sigma^2 = 1$ by the likelihood ratio test.

   (b) Let X_1, X_2, X_3, X_4, X_5 be a i.i.d random variables with p.d.f

   $$f(x) \quad = \quad \frac{1}{\theta}e^{-\frac{x}{\theta}}, \quad x \geq 0, \theta > 0.$$

   Find the most powerful critical region for testing $H_0 : \theta = 1$ against $\theta = 2$ at $\alpha = 0.05$.

   (c) Let X be a random variable of continuous type with p.d.f.

   $$f(x) = \left(\frac{\theta}{x}\right)\left(\frac{3}{x}\right)^{\theta}, \; x > 3, \theta > 0.$$

   Based on the single observation X, the MP test of size $\alpha = 0.1$ for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$, rejects H_0 if $X < k$. Find the value of k.

   (d) Let X be a random variable of continuous type with p.d.f. $f(x)$. Based on the single observation X, the MP test of size $\alpha = 0.1$, find the MP critical region for testing $H_0 : 2x, 0 < x < 1$ against $H_1 : f(x) = 4x^3, 0 < x < 1$. Also find the power of the test.

Problem on Estimation

2. Let X be a random variable of continuous type with p.m.f.

   $$f(x) = (1-p)^{x-1}p, x = 1, 2, 3, \cdots$$

   Find the MLE of $\frac{1}{p(1-p)}$. Is it unbiased?

Problem on SPRT

3. Obtain SPRT for the fraction defective under a manufacturing process with $H_0 : p_0 = 0.04$ against $H_1 : p_1 = 0.08$ with the strength of $(0.15, 0.25)$. What will be your decision about the process if the result of a sampling inspection procedure have the following result ?

m: 1	2	3	4	5	6	7	8	9	10
x: 0	0	0	1	0	0	1	0	1	1

Problem on power curve

4. Random sample of size 10 is drawn from a normal distribution with unknown mean μ and known standard deviation $\sigma = 2$. It is desired to test

$$(i) H_0 : \mu = \mu_0 = 0 \text{ against } H_1 : \mu > \mu_0$$
$$(ii) H_0 : \mu = \mu_0 = 0 \text{ against } H_2 : \mu < \mu_0$$
$$(iii) H_0 : \mu = \mu_0 = 0 \text{ against } H_3 : \mu \neq \mu_0$$

Find the power function for each alternatives and draw the power curve for $\mu = -0.5, -1, -1.5, 0.5, 1, 1.5$.

Solutions

1. **Most Powerful Test**

(b) Given that
(i) $f(x) = \left(\frac{\theta}{x}\right)\left(\frac{3}{x}\right)^{\theta}$, $x > 3, \theta > 0$
(ii) $H_0 : \theta = 1$ against $H_1 : \theta = 2$
(iii) MP critical region is $W = \{X | X < k\}$
(iv) $\alpha = 0.1$
By Neyman-Pearson lemma
$P(X \in W | H_0) = \alpha$
$\Rightarrow P(X < k | \theta = 1) = 0.1$
$\Rightarrow \int_3^k \left(\frac{3}{x^2}\right) dx$
$\Rightarrow 3\left[-\frac{1}{x}\right]_3^k = 0.1$
$\Rightarrow 1 - \frac{3}{k} = 0.1$
$\Rightarrow k = \frac{10}{3}$

(c) Given that
$H_0 : f(x) = 2x, 0 < x < 1,$
$H_1 : f(x) = 4x^3, 0 < x < 1,$

and $\alpha = 0.1$

By Neyman-Pearson lemma the most powerful critical region is found by $\frac{f_{H_1}(x)}{f_{H_0}(x)} > k$

$\frac{4x^3}{2x} > k$

$2x^2 > k$

$x^2 > k_1$

$x > k_2$

Now k_2 is found from the size condition, that is $P(X \in W|H_0) = \alpha$

$\Rightarrow P(X > k_2|H_0) = 0.1$

$\Rightarrow \int_1^{k_2} 2xdx = 0.1$

$\Rightarrow [x^2]_1^{k_2} = 0.1$

$\Rightarrow 1 - {k_2}^2 = 0.1$

$\Rightarrow {k_2}^2 = 0.9$

$\Rightarrow {k_2}^2 = \frac{9}{10}$

$\Rightarrow k_2 = \frac{3}{\sqrt{10}}$

Now the power of the test is

$P(X \in W|H_1)$

$= P(X > \frac{3}{\sqrt{10}}|H_1)$

$= \int_{\frac{3}{\sqrt{10}}}^{1} 4x^3dx$

$= [x^4]_{\frac{3}{\sqrt{10}}}^{1}$

$= 1 - \frac{81}{100}$

$= \frac{19}{100}$

Estimation

2. Given that

$$f(x) = (1-p)^{x-1}p, x = 1, 2, 3, \cdots$$

Therefore the likelihood function is given by

$L(p) = (1-p)^{x-1}p$, since the single observation is given,

taking log_e on both sides we have

$ln(p) = (x-1)ln(1-p) + lnp$

$\Rightarrow \frac{dlnL(p)}{dp} = -\frac{x-1}{1-p} + \frac{1}{p} \cdots (i)$

$\Rightarrow \frac{d^2lnL(p)}{dp^2} = -\frac{x-1}{(1-p)^2} - \frac{1}{p^2} \cdots (ii)$

Now $\frac{dlnL(p)}{dp} = 0$

$\Rightarrow \frac{x-1}{p-1} = \frac{1}{p}$

$\Rightarrow px - p = 1 - p$
$\Rightarrow px = 1$
$\Rightarrow p = \frac{1}{x}$
Since from (ii) we get $\frac{d^2 lnL(p)}{dp^2} < 0$, so the maximum likelihood estimator is $\hat{p}_{MLE} = \frac{1}{x}$.
By the invaiance property of maximum likelihood estimator of of $\frac{1}{p(1-p)}$ is

$$\frac{1}{\hat{p}_{MLE}(1-\hat{p}_{MLE})} = \frac{1}{\frac{1}{x}(1-\frac{1}{x})} = \frac{x^2}{(x-1)}$$

Since $E\left(\frac{x^2}{x-1}\right) \neq \frac{1}{p(1-p)}$ (check this), $\frac{1}{\hat{p}_{MLE}(1-\hat{p}_{MLE})}$ is not unbiased estimator of $\frac{1}{p(1-p)}$.

SPRT

3. Given that the strength of the SPRT is $(\alpha_0, \alpha_1) = (0.15, 0.25)$. Our objective is to test the hypothesis $H_0 : p = 0.04$ against $H_1 : p = 0.08$. It is also given that the experiment is Bernoullian type, so we assume

$$X_i = \begin{cases} 1, & \text{if i-th unit is defective,} \\ 0, & \text{i-th unit is not defective.} \end{cases}$$

So under $H_0 : P(X_i = 1) = 0.04$ and $P(X_i = 0) = 0.96$ and under $H_1 : P(X_i = 1) = 0.08$ and $P(X_i = 0) = 0.92$. For practical purpose we assume that

$$lnL_0 = ln\left(\frac{\alpha_1}{1-\alpha_0}\right) \text{ and } lnL_1 = ln\left(\frac{1-\alpha_1}{\alpha_0}\right).$$

From the theory of SPRT we have

(i)Reject H_0 if $\sum_{i=1}^m X_i \geq \frac{ln\left(\frac{1-\alpha_1}{\alpha_0}\right) - mln\left(\frac{1-p_1}{1-p_0}\right)}{ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)} = r_m(\text{say}).$

(ii)Aeject H_0 if $\sum_{i=1}^m X_i \leq \frac{ln\left(\frac{\alpha_0}{1-\alpha_1}\right) - mln\left(\frac{1-p_1}{1-p_0}\right)}{ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)} = a_m(\text{say}).$

(iii)Continue the sampling if $\frac{ln\left(\frac{\alpha_0}{1-\alpha_1}\right) - mln\left(\frac{1-p_1}{1-p_0}\right)}{ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)} < \sum_{i=1}^m X_i < \frac{ln\left(\frac{1-\alpha_1}{\alpha_0}\right) - mln\left(\frac{1-p_1}{1-p_0}\right)}{ln\left(\frac{p_1(1-p_0)}{p_0(1-p_1)}\right)}$

that is if $a_m < \sum_{i=1}^m X_i < r_m$

m	$\sum_{i=1}^{m} X_i$	a_m	r_m	conclusion
1	0	-1.62	2.24	continued
2				
3				
4				
5				
6				
7				
8				
9				
10				

Calculate the remaining part of the table by own and write the conclusion.

Power curve

4. Consider the testing problem $H_0 : \mu = \mu_0 = 0$ against $H_1 : \mu > \mu_0$, the Mp critical region is given by

$$W_1 = \{\underline{x} | c < \bar{x} < \infty\}$$

Now first find c from the size condition, that is
$P(\bar{x} > c | H_0) = 0.05$
$\Rightarrow P(\frac{\bar{x}-\mu_0}{2/\sqrt{10}} > \frac{c-\mu_0}{2/\sqrt{10}}) = 0.05$
$\Rightarrow P(\frac{\bar{x}-0}{2/\sqrt{10}} > \frac{c-0}{2/\sqrt{10}}) = 0.05$
$\Rightarrow P(\tau > \frac{\sqrt{10}c}{2}) = 0.05$, where $\tau = \frac{\bar{x}}{2/\sqrt{10}} \sim N(0,1)$
$\Rightarrow 1 - \Phi(\frac{\sqrt{10}c}{2}) = 0.05$
$\Rightarrow \Phi(\frac{\sqrt{10}c}{2}) = 0.95 \Rightarrow \frac{\sqrt{10}c}{2} = 1.64$ (from table)
$\Rightarrow c = 1.04$
Now the power function is given by $P(\bar{x} > c | H_1)$
$P\left(\frac{\sqrt{10}(\bar{x}-\mu)}{2} > \frac{\sqrt{10}(1.04-\mu)}{2}\right)$
$= 1 - \Phi(\frac{\sqrt{10}(1.04-\mu)}{2}) = P_\mu(W)$, (say).
So for drawing the power curve calculate the following table (using the statistical table for normal distribution)

μ	$P_\mu(W)$
-1.5	0.0075
-1	0.006
-0.5	0
0.5	0.1976
1	0.4840
1.5	0.7670

Students please complete the remaining part of the problem.